Exercise sheet 2

Tuesday 6th January 2026

Minimal surfaces in hyperbolic manifolds

Exercise 1: Minimal graphs in the upper half-space and parabolic invariant solutions

1. (*Not too hard, if one has already done Exercise 3 of Sheet 1*) In the upper half-space
$$\mathbb{H}^3 = \left(\{(x,y,z) \,|\, z > 0\}, \frac{dx^2 + dy^2 + dz^2}{z^2} \right) ,$$
let $\Sigma \subset \mathbb{H}^3$ be the graph of a function $u : \Omega \to (0, +\infty)$ over the (x,y)-plane. Show that Σ is minimal if and only if
$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} + \frac{2(1 + u_x^2 + u_y^2)}{u} = 0 \ .$$
2. (*Easy*) Write down an ODE which is equivalent to finding a surface Σ as above that is invariant under a parabolic one-parameter group of isometries $(x,y,z) \mapsto (x, y+t, z)$.
3. (*Medium*) Show that the maximal time of existence of such ODE is finite.
4. (*Harder*) Show that there exists a unique complete minimal parabolic-invariant graph, up to isometry. What is its boundary at infinity?

Exercise 2: Geometric maximum principle

1. (*Easy*) Let $\Sigma \subset \mathbb{R}^3$ be the graph of a function $u : \Omega \to \mathbb{R}$, for Ω an open subset of $\mathbb{R}^2$. Suppose that $du_p = 0$. Prove that the second fundamental form of Σ at $p = (x,y,z)$, computed with respect to the upward normal vector field, equals the (Euclidean) Hessian of f at (x,y).
2. (*Easy*) Deduce the following statement: if two embedded surfaces Σ_- and Σ_+ are equal up to order one at p (i.e. p is in both Σ_- and Σ_+, and $T_p\Sigma_- = T_p\Sigma_+$) and Σ_+ lies above Σ_- in a neighbourhood of p, then the mean curvatures $H_\pm$ of $\Sigma_\pm$ satisfy $H_-(p) \leq H_+(p)$.
3. (*Medium*) Generalize to any ambient Riemannian manifold. [Hint: use normal coordinates centered at p.]
4. (*Optional, and intentionally vague*) Look up and apply the strong maximum principle to obtain the following additional statement: if both Σ_- and Σ_+ are minimal and connected, then $\Sigma_- = \Sigma_+$.
5. (*Medium*) Prove that there is no closed (i.e. compact without boundary) minimal surface in $\mathbb{R}^3$ or $\mathbb{H}^3$.
6. (*Medium*) Explain why the proof of the previous point fails in $\mathbb{S}^3$ (see Exercise 2 of Sheet 1 for counterexamples).

Exercise 3: Almost-Fuchsian manifolds

1. (*Medium*) Let Σ be a minimal surface in a complete hyperbolic manifold such that $\|I\!I_\Sigma\|^2 \leq 2$. Let Σ_t (for $t \in \mathbb{R}$) be the set
$$\Sigma_t = \{\exp_p(tN(p)) \,|\, p \in \Sigma\}$$
where N is the unit normal vector of Σ. Prove that Σ_t is an immersed surface and that the pull-back to Σ of its first and second fundamental forms are
$$I_t = I((\cos t)\mathrm{id} + (\sin t)B, (\cos t)\mathrm{id} + (\sin t)B) \quad \text{and} \quad I\!I_t = I((\cos t)B - (\sin t)\mathrm{id}, (\cos t)B - (\sin t)\mathrm{id})$$
[Hint: it is convenient to use the hyperboloid model of $\mathbb{H}^3$ since the exponential map is particularly simple.]

2. (*Easy*) Show that, if $\|II_\Sigma\|^2 < 2$, then Σ_t is convex for $|t| > c$, for c a constant to be computed.

3. (*Easy*) Show that the mean curvature of Σ_t is negative for $t > 0$ and positive for $t < 0$ — that is, the mean curvature vector always points towards Σ.

4. (*Medium*) Apply the geometric maximum principle to show that if a complete three-manifold M admits a foliation by closed surfaces where one leaf (say Σ) is minimal and all the other leaves have mean curvature vector pointing towards Σ, then Σ is the unique closed minimal surface in M.

5. (*Easy, at this point*) Conclude that a weakly almost-Fuchsian manifold — that is, a complete hyperbolic manifold homeomorphic to $S \times \mathbb{R}$ admitting a closed minimal surface Σ with $\|II_\Sigma\|^2 \leq 2$ — has a unique closed minimal surface.

Comment: a conjecture often attributed to Thurston asserts that every almost-Fuchsian manifold admits a foliation where all the leaves have *constant* mean curvature, where (as in point 4) one leaf is the minimal surface and the mean curvature vector of all the other leaves point towards the minimal surface.